

POLYNOMIAL PARTITIONING FOR SEVERAL SETS OF VARIETIES

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ABSTRACT. We give a new, systematic proof for a recent result of Larry Guth and thus also extend the result to a setting with several families of varieties: For any integer $D \geq 1$ and any collection of sets $\Gamma_1, \dots, \Gamma_j$ of low-degree k -dimensional varieties in \mathbb{R}^n there exists a non-zero polynomial $p \in \mathbb{R}[X_1, \dots, X_n]$ of degree at most D so that each connected component of $\mathbb{R}^n \setminus Z(p)$ intersects $O(jD^{k-n}|\Gamma_i|)$ varieties of Γ_i , simultaneously for every $1 \leq i \leq j$. For $j = 1$ we recover the original result by Guth. Our proof, via an index calculation in equivariant cohomology, shows how the degrees of the polynomials used for partitioning are dictated by the topology, namely by the Euler class being given in terms of a top Dickson polynomial.

1. INTRODUCTION

The celebrated work [5] by Larry Guth and Nets Hawk Katz on the Erdős distinct distances problem in the plane brought to light the following beautiful partitioning result:

Theorem 1.1 (Guth and Katz 2015 [5, Thm. 4.1]). *Let X be a finite set of points in \mathbb{R}^n , and let $D \geq 1$ be an integer. Then there exists a non-zero polynomial $p \in \mathbb{R}[X_1, \dots, X_n]$ of degree at most D such that each connected component of the complement $\mathbb{R}^n \setminus Z(p)$ contains at most $C_n D^{-n} |X|$ points of X , where C_n is a constant that may depend on n .*

Here $Z(p)$ denotes the set of zeroes in \mathbb{R}^n of the polynomial p , that is

$$Z(p) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : p(x_1, \dots, x_n) = 0\}.$$

In his recent paper [4], Guth used equivariant topology to prove the following extended polynomial partitioning result.

Theorem 1.2 (Guth, 2015 [4, Thm. 0.3]). *Let Γ be a finite set of k -dimensional varieties in \mathbb{R}^n , each of them defined by at most m polynomial equations of degree at most d . Then for any $D \geq 1$ there exists a non-zero polynomial $p \in \mathbb{R}[X_1, \dots, X_n]$ of degree at most D such that each connected component of the complement $\mathbb{R}^n \setminus Z(p)$ intersects at most $C(d, m, n) D^{k-n} |\Gamma|$ varieties in Γ , where $C(d, m, n)$ is a constant that may depend on the parameters d , m , and n .*

In this paper, based on the set-up from the proof of the previous theorem and the use of the Fadell–Husseini index theory [3] for the proof of a necessary Borsuk–Ulam type theorem, we make the next extension step by proving the following “colored” generalization of Theorem 1.2.

Theorem 1.3. *Let $j \geq 1$ be an integer. For $1 \leq i \leq j$, let Γ_i be a finite set of k_i -dimensional varieties in \mathbb{R}^n , each of them defined by at most m_i polynomial equations of degree at most d_i . Then for any $D \geq 1$ there exists a non-zero polynomial $p \in \mathbb{R}[X_1, \dots, X_n]$ of degree at most D such that each connected component of the complement $\mathbb{R}^n \setminus Z(p)$ for every $1 \leq i \leq j$ intersects at most $C(d_i, m_i, n) j D^{k_i-n} |\Gamma_i|$ varieties in Γ_i , where $C(d_i, m_i, n)$ is a constant that may depend on parameters d_i , m_i , and n .*

In a concrete example, this says the following: There are constants $C_1 = C(1, 2, 3)$ and $C_2 = C(1, 3, 3)$ such that if we have (large) collections Γ_1 of red lines and Γ_2 of blue points in \mathbb{R}^3 , then for every $D \geq 1$ there is a nonzero polynomial $p(x, y, z) \in \mathbb{R}[x, y, z]$ of degree at most D such that each connected component of $\mathbb{R}^3 \setminus Z(p)$ meets at most $2C_1 \frac{|\Gamma_1|}{D^2}$ red lines, and at most $2C_2 \frac{|\Gamma_2|}{D^3}$ blue points. (This is the special case when we have $j = 2$ families of varieties in \mathbb{R}^3 , so $n = 3$, where the first family consists of lines, so $k_1 = 1$ and e.g. $m_1 = 2$, $d_1 = 1$, and the second one of points, so $k_2 = 0$, $m_2 = 3$, $d_2 = 1$.)

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Note added in revision. In his recent paper [8], Ben Yang used Theorem 1.3 to obtain new upper bounds on the number of generalized joints for (possibly higher-dimensional) varieties.

2. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 will have several separate components that at the end of the proof merge into the final argument. We also rely on two particular results by József Solymosi and Terence Tao [7, Thm. A.2] and by Guth [4, Lemma 3.1].

2.1. Let P_n^δ be the vector space of polynomials in n variables of degree at most δ with real coefficients,

$$P_n^\delta = \{p \in \mathbb{R}[X_1, \dots, X_n] : \deg p \leq \delta\}.$$

The dimension of this vector space is $\dim P_n^\delta = \binom{\delta+n}{n} > \frac{\delta^n}{n!}$. For every integer $\ell \geq 1$ choose the smallest integer δ_ℓ with the property that

$$j2^{\ell-1} \leq \frac{\delta_\ell^n}{n!} < j2^n 2^{\ell-1},$$

or equivalently

$$(n!)^{\frac{1}{n}} j^{\frac{1}{n}} 2^{\frac{\ell-1}{n}} \leq \delta_\ell < 2(n!)^{\frac{1}{n}} j^{\frac{1}{n}} 2^{\frac{\ell-1}{n}}. \quad (1)$$

In particular, $\dim P_n^{\delta_\ell} > j2^{\ell-1}$. Next, let s be the smallest integer such that

$$\sum_{\ell=1}^s \delta_\ell \leq D < \sum_{\ell=1}^{s+1} \delta_\ell. \quad (2)$$

The inequalities (1) and (2) imply that

$$D < \sum_{\ell=1}^{s+1} \delta_\ell < 2(n!)^{\frac{1}{n}} j^{\frac{1}{n}} \sum_{\ell=1}^{s+1} 2^{\frac{\ell-1}{n}} = 2(n!)^{\frac{1}{n}} j^{\frac{1}{n}} \frac{2^{\frac{s+1}{n}} - 1}{2^{\frac{1}{n}} - 1} < \frac{2(n!)^{\frac{1}{n}}}{2^{\frac{1}{n}} - 1} j^{\frac{1}{n}} 2^{\frac{s+1}{n}} = \frac{2^{1+\frac{1}{n}} (n!)^{\frac{1}{n}}}{2^{\frac{1}{n}} - 1} j^{\frac{1}{n}} 2^{\frac{s}{n}}. \quad (3)$$

Consequently, the inequality (3) gives

$$D^n < \frac{2^{n+1} n!}{(2^{\frac{1}{n}} - 1)^n} j 2^s \implies \frac{1}{2^s} < \frac{2^{n+1} n!}{(2^{\frac{1}{n}} - 1)^n} \frac{j}{D^n} = C_n \frac{j}{D^n}, \quad (4)$$

where C_n depends only on n .

2.2. For every $1 \leq \ell \leq s$ we have that $\dim P_n^{\delta_\ell} \geq j2^{\ell-1} + 1$. Let V_ℓ denote an arbitrary vector subspace of $P_n^{\delta_\ell}$ of dimension $j2^{\ell-1} + 1$. The unit sphere $S(V_\ell)$ in the vector space V_ℓ is equipped with the free $\mathbb{Z}/2 = \langle \omega_\ell \rangle$ action given by $\omega_\ell \cdot p = -p$ for $p \in S(V_\ell)$.

We will use the product space

$$Y := \prod_{\ell=1}^s S(V_\ell) \cong \prod_{\ell=1}^s S^{j2^{\ell-1}}.$$

The elementary abelian group $(\mathbb{Z}/2)^s = \langle \omega_1, \dots, \omega_s \rangle$ acts on Y componentwise, that is, for $1 \leq \ell \leq s$ and $(p_1, \dots, p_\ell, \dots, p_s) \in Y$ the generator ω_ℓ acts as follows:

$$\omega_\ell \cdot (p_1, \dots, p_\ell, \dots, p_s) = (p_1, \dots, -p_\ell, \dots, p_s). \quad (5)$$

2.3. Consider the vector space $\mathbb{R}^{(\mathbb{Z}/2)^s}$ and the vector subspace of codimension 1 given by

$$U_s = \left\{ (y_\alpha)_{\alpha \in (\mathbb{Z}/2)^s} \in \mathbb{R}^{(\mathbb{Z}/2)^s} : \sum_{\alpha \in (\mathbb{Z}/2)^s} y_\alpha = 0 \right\}.$$

We introduce the following action of $(\mathbb{Z}/2)^s$ on $\mathbb{R}^{(\mathbb{Z}/2)^s}$: The element $(\beta_1, \dots, \beta_s) \in (\mathbb{Z}/2)^s$ acts on the vector $(y_\alpha)_{\alpha \in (\mathbb{Z}/2)^s} \in \mathbb{R}^{(\mathbb{Z}/2)^s}$ by acting on its index set

$$(\beta_1, \dots, \beta_s) \cdot (\alpha_1, \dots, \alpha_s) = (\beta_1 + \alpha_1, \dots, \beta_s + \alpha_s), \quad (6)$$

where the addition is assumed to be in $\mathbb{Z}/2$. With respect to the introduced action the vector subspace U_s is a $(\mathbb{Z}/2)^s$ -subrepresentation of $\mathbb{R}^{(\mathbb{Z}/2)^s}$ of dimension $2^s - 1$.

2.4. Any non-constant polynomial $p \in \mathbb{R}[X_1, \dots, X_n]$ determines two disjoint open regions in \mathbb{R}^n , possibly one of them empty, which we denote by

$$D_p^0 = \{(x_1, \dots, x_n) \in \mathbb{R}^n : p(x_1, \dots, x_n) > 0\} \quad \text{and} \quad D_p^1 = \{(x_1, \dots, x_n) \in \mathbb{R}^n : p(x_1, \dots, x_n) < 0\}.$$

Thus $\mathbb{R}^n \setminus Z(p) = D_p^0 \cup D_p^1$ and $D_p^0 \cap D_p^1 = \emptyset$.

Let $(p_1, \dots, p_s) \in Y$ be an ordered tuple of polynomials in Y , and let $\alpha = (\alpha_1, \dots, \alpha_s) \in (\mathbb{Z}/2)^s = \{0, 1\}^s$. The *sign pattern domain* determined by the tuple (p_1, \dots, p_s) and by the element $\alpha \in (\mathbb{Z}/2)^s$ is the intersection of open regions

$$\mathcal{O}_\alpha^{(p_1, \dots, p_s)} = D_{p_1}^{\alpha_1} \cap \dots \cap D_{p_s}^{\alpha_s}.$$

An sign pattern domain can be empty. Moreover

$$\mathbb{R}^n \setminus Z(p_1 \cdots p_s) = \bigcup_{\alpha \in (\mathbb{Z}/2)^s} \mathcal{O}_\alpha^{(p_1, \dots, p_s)}, \quad (7)$$

where the union is disjoint union. Observe that $\deg(p_1 \cdots p_s) \leq \sum_{\ell=0}^s \delta_\ell \leq D$. Furthermore, the sign pattern domains $\mathcal{O}_\alpha^{(p_1, \dots, p_s)}$ are unions of connected components of the complement $\mathbb{R}^n \setminus Z(p_1 \cdots p_s)$.

2.5. For every $\alpha \in (\mathbb{Z}/2)^s$ and every variety $\gamma \subset \mathbb{R}^n$ we define the function $\phi_{\alpha, \gamma} : Y \rightarrow \mathbb{R}$ by

$$\phi_{\alpha, \gamma}(p_1, \dots, p_s) = \begin{cases} 1, & \text{if } \mathcal{O}_\alpha^{(p_1, \dots, p_s)} \cap \gamma \neq \emptyset \\ 0, & \text{if } \mathcal{O}_\alpha^{(p_1, \dots, p_s)} \cap \gamma = \emptyset, \end{cases}$$

where $(p_1, \dots, p_s) \in Y$. The functions $\phi_{\alpha, \gamma}$ are not continuous, but as Guth showed in [4, Lemma 3.1], they can be approximated by sequences of continuous functions:

Lemma 3.1. *For $\varepsilon > 0$, $\gamma \subset \mathbb{R}^n$ and $\alpha \in (\mathbb{Z}/2)^s$, we define functions $\phi_{\alpha, \gamma}^\varepsilon : Y \rightarrow \mathbb{R}$ with the following properties.*

- (1) *The functions $\phi_{\alpha, \gamma}^\varepsilon : Y \rightarrow \mathbb{R}$ are continuous.*
- (2) *$0 \leq \phi_{\alpha, \gamma}^\varepsilon \leq 1$.*
- (3) *If $\mathcal{O}_\alpha^{(p_1, \dots, p_s)} \cap \gamma = \emptyset$, then $\phi_{\alpha, \gamma}^\varepsilon = 0$.*
- (4) *If $\varepsilon_i \rightarrow 0$ and $(p_1^i, \dots, p_s^i) \rightarrow (p_1, \dots, p_s)$ in Y , and $\mathcal{O}_\alpha^{(p_1, \dots, p_s)} \cap \gamma \neq \emptyset$, then*

$$\lim_{i \rightarrow \infty} \phi_{\alpha, \gamma}^{\varepsilon_i}(p_1^i, \dots, p_s^i) = 1.$$

In other words, $\phi_{\alpha, \gamma}(p_1, \dots, p_s) \leq \liminf_{i \rightarrow \infty} \phi_{\alpha, \gamma}^{\varepsilon_i}(p_1^i, \dots, p_s^i)$.

In order to simplify the presentation we postpone the typical compactness argument applied to Y to the last step in the proof (2.8) and for now continue to work with the functions $\phi_{\alpha, \gamma}$ as if they were continuous.

2.6. Let $1 \leq i \leq j$ be fixed. By assumption Γ_i is a finite set of k_i -dimensional varieties in \mathbb{R}^n , each defined by at most m_i polynomial equations of degree at most d_i . Consider the following map from the space Y to the representation $U_s^{\oplus j}$ associated to the collection of finite sets of varieties $\Gamma_1, \dots, \Gamma_j$ given in the theorem:

$$\begin{aligned} \Phi : Y &\rightarrow U_s^{\oplus j}, \\ (p_1, \dots, p_s) &\mapsto \left(\left(\sum_{\gamma \in \Gamma_i} \phi_{\alpha, \gamma}(p_1, \dots, p_s) - \frac{1}{2^s} \sum_{\beta \in (\mathbb{Z}/2)^s} \sum_{\gamma \in \Gamma_i} \phi_{\beta, \gamma}(p_1, \dots, p_s) \right)_{\alpha \in (\mathbb{Z}/2)^s} \right)_{i \in \{1, \dots, j\}}. \end{aligned} \quad (8)$$

The sum $\sum_{\gamma \in \Gamma_i} \phi_{\alpha, \gamma}(p_1, \dots, p_s)$ counts the number of varieties in Γ_i that intersect sign pattern domain $\mathcal{O}_\alpha^{(p_1, \dots, p_s)}$. The map Φ is continuous and $(\mathbb{Z}/2)^s$ -equivariant with respect to the actions given by (5) and (6), assuming the diagonal action on the direct sum $U_s^{\oplus j}$.

Let us assume that $\Phi^{-1}(0) \neq \emptyset$, and pick $(p_1, \dots, p_s) \in \Phi^{-1}(0)$. Then each of 2^s sign pattern domains $\mathcal{O}_\alpha^{(p_1, \dots, p_s)}$, $\alpha \in (\mathbb{Z}/2)^s$, determined by the tuple (p_1, \dots, p_s) in Y intersects the same number of varieties in the set Γ_i , for every $1 \leq i \leq j$. We use the following result of Solymosi and Tao [7, Thm. A.2], stated as in [4, Thm. 0.2]:

Theorem A.2. *Suppose that γ is a k -dimensional variety in \mathbb{R}^n defined by m polynomial equations each of degree at most d . If p is a polynomial of degree at most D , then γ intersects at most $C'(d, m, n)D^k$ different connected components of $\mathbb{R}^n \setminus Z(p)$, where $C'(d, m, n)$ is a constant that may depend on the parameters d_i , m_i , and n .*

Hence, each variety $\gamma \in \Gamma_i$ intersects at most $C'(d_i, m_i, n)D^{k_i}$ connected components of the complement $\mathbb{R}^n \setminus Z(p_1 \cdots p_s)$. Since each sign pattern domain $\mathcal{O}_\alpha^{(p_1, \dots, p_s)}$ is a disjoint union of connected components of $\mathbb{R}^n \setminus Z(p_1 \cdots p_s)$, each variety $\gamma \in \Gamma_i$ intersects at most $C'(d_i, m_i, n)D^{k_i}$ sign pattern domains. As we are looking at a point $(p_1, \dots, p_s) \in \Phi^{-1}(0)$, where we get the same number of varieties $\gamma \in \Gamma_i$ intersecting each sign pattern domain $\mathcal{O}_\alpha^{(p_1, \dots, p_s)}$, this number is at most $\frac{1}{2^s}|\Gamma_i|C'(d_i, m_i, n)D^{k_i}$. The inequality (4) implies that

$$\frac{1}{2^s}|\Gamma_i|C'(d_i, m_i, n)D^{k_i} < C_n C'(d_i, m_i, n)jD^{k_i-n}|\Gamma_i|.$$

Each connected component of $\mathbb{R}^n \setminus Z(p_1 \cdots p_s)$ is contained in a unique sign pattern domain, and therefore the number of varieties $\gamma \in \Gamma_i$ intersecting a connected component of $\mathbb{R}^n \setminus Z(p_1 \cdots p_s)$ cannot exceed

$$C(d_i, m_i, n) \cdot jD^{k_i-n}|\Gamma_i|,$$

where $C(d_i, m_i, n)$ is a constant that may depend on the parameters d_i , m_i , and n . This concludes the proof of Theorem 1.3, except that it remains to be verified that the map Φ indeed has a zero.

2.7. We still need to prove that the $(\mathbb{Z}/2)^s$ -equivariant map $\Phi: Y \longrightarrow U_s^{\oplus j}$, defined in (8), has a zero. Indeed, in the spirit of the usual resolution of “configuration space/test map schemes” for discrete geometry problems [6], we will show that there is no continuous $(\mathbb{Z}/2)^s$ -equivariant map $Y \longrightarrow U_s^{\oplus j}$ at all that avoids zero.

Let us assume to the contrary that there is such a map, then this induces a continuous $(\mathbb{Z}/2)^s$ -equivariant map $Y \longrightarrow S(U_s^{\oplus j})$, where $S(U_s^{\oplus j})$ denotes the unit sphere in the vector spaces $U_s^{\oplus j}$. Using the Fadell–Husseini ideal-valued index theory [3] for the group $(\mathbb{Z}/2)^s$ and \mathbb{F}_2 coefficients, we will prove that such an equivariant map cannot exist, obtaining the required contradiction.

The cohomology of the group $(\mathbb{Z}/2)^s$ with \mathbb{F}_2 coefficients is given by $H^*((\mathbb{Z}/2)^s; \mathbb{F}_2) \cong \mathbb{F}_2[u_1, \dots, u_s]$, where $\deg(u_i) = 1$ and the variable u_i corresponds to the generator ω_i , for $1 \leq i \leq s$. According to [3, Ex. 3.3],

$$\text{Index}_{(\mathbb{Z}/2)^s}(Y; \mathbb{F}_2) = \langle u_1^{j+1}, u_2^{2j+1}, u_3^{4j+1}, \dots, u_s^{j2^{s-1}+1} \rangle.$$

Furthermore, from [3, Prop. 3.13] (see also [2, Prop. 3.12 and Prop. 3.13]) we have that

$$\text{Index}_{(\mathbb{Z}/2)^s}(S(U_s^{\oplus j}); \mathbb{F}_2) = \left\langle \left(\prod_{(\alpha_1, \dots, \alpha_s) \in (\mathbb{Z}/2)^s \setminus \{0\}} (\alpha_1 u_1 + \dots + \alpha_s u_s) \right)^j \right\rangle.$$

Since a continuous $(\mathbb{Z}/2)^s$ -equivariant map $Y \longrightarrow S(U_s^{\oplus j})$ exists, the basic property of the Fadell–Husseini index [3, Sec. 2] implies that

$$\text{Index}_{(\mathbb{Z}/2)^s}(S(U_s^{\oplus j}); \mathbb{F}_2) \subseteq \text{Index}_{(\mathbb{Z}/2)^s}(Y; \mathbb{F}_2),$$

and consequently that

$$\left(\prod_{(\alpha_1, \dots, \alpha_s) \in (\mathbb{Z}/2)^s \setminus \{0\}} (\alpha_1 u_1 + \dots + \alpha_s u_s) \right)^j \in \langle u_1^{j+1}, u_2^{2j+1}, u_3^{4j+1}, \dots, u_s^{j2^{s-1}+1} \rangle. \quad (9)$$

The polynomial

$$q := \prod_{(\alpha_1, \dots, \alpha_s) \in (\mathbb{Z}/2)^s \setminus \{0\}} (\alpha_1 u_1 + \dots + \alpha_s u_s) \in \mathbb{F}_2[u_1, \dots, u_s].$$

is the Dickson polynomial of maximal degree [1, Sec. III.2]. It can be presented in the form

$$q = \sum_{\pi \in \mathfrak{S}_s} u_{\pi(1)}^{2^{s-1}} u_{\pi(2)}^{2^{s-2}} \cdots u_{\pi(s)}^{2^0}.$$

Now the j -th power of the Dickson polynomial q^j can be decomposed as follows

$$\begin{aligned} q^j &= \left(\prod_{(\alpha_1, \dots, \alpha_s) \in (\mathbb{Z}/2)^s \setminus \{0\}} (\alpha_1 u_1 + \dots + \alpha_s u_s) \right)^j = \left(\sum_{\pi \in \mathfrak{S}_s} u_{\pi(1)}^{2^{s-1}} u_{\pi(2)}^{2^{s-2}} \cdots u_{\pi(s)}^{2^0} \right)^j \\ &= (u_s^{j2^{s-1}} u_{s-1}^{j2^{s-2}} \cdots u_2^{2j} u_1^j) + \text{Rest}, \end{aligned}$$

where “Rest” denotes a polynomial that does not contain the monomial $u_s^{j2^{s-1}} u_{s-1}^{j2^{s-2}} \cdots u_2^{2j} u_1^j$. Hence,

$$q^j \notin \langle u_1^{j+1}, u_2^{2j+1}, u_3^{4j+1}, \dots, u_s^{j2^{s-1}+1} \rangle, \quad (10)$$

in contradiction to relation (9).

2.8. We just proved that there cannot be any continuous $(\mathbb{Z}/2)^s$ -equivariant map $Y \rightarrow S(U_s^{\oplus j})$. Therefore, every continuous $(\mathbb{Z}/2)^s$ -equivariant map $Y \rightarrow U_s^{\oplus j}$ has a zero.

More is true: Since Y is compact, every $(\mathbb{Z}/2)^s$ -equivariant map $Y \rightarrow U_s^{\oplus j}$, not necessarily continuous, which is a limit of a sequence of continuous $(\mathbb{Z}/2)^s$ -equivariant maps $Y \rightarrow U_s^{\oplus j}$, will also have a zero. Indeed, let $\Psi := \lim_{i \rightarrow \infty} \Psi^i$ where $\Phi^i: Y \rightarrow U_s^{\oplus j}$ are continuous $(\mathbb{Z}/2)^s$ -equivariant maps. Since the maps Ψ^i are continuous, for every i there exists a $y_i \in Y$ such that $\Psi^i(y_i) = 0$. The compactness of Y yields the existence of a converging subsequence $\lim_{i \rightarrow \infty} y_{k(i)} = y \in Y$, where $k: \mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing function. Thus $\Psi(y) = \lim_{i \rightarrow \infty} \Psi^{k(i)}(y_{k(i)}) = 0$.

We have proved in (2.5) that the maps $\phi_{\alpha, \gamma}$, used for the definition of the map Φ , are limits of sequences of continuous maps. Thus the $(\mathbb{Z}/2)^s$ -equivariant map $\Phi: Y \rightarrow U_s^{\oplus j}$, even not continuous by construction, still has a zero. The proof of Theorem 1.3 is now complete.

Remark. The choice of the degree bounds δ_ℓ for the polynomials p_ℓ used for partitioning, and consequently of the vector spaces V_ℓ etc., which in the special case $j = 1$ already appeared in Guth's work [4], can now be seen as very natural if one tries to show that at least one monomial in the power of the Dickson polynomial q does not belong to the index of the configuration space Y of s -tuples of polynomials (p_1, \dots, p_s) , and thus to obtain a contradiction in (10).

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